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1976 J. Phys. A: Math. Gen. 9 229

(http://iopscience.iop.org/0305-4470/9/2/007)

J. Phys. A: Math. Gen., Vol. 9, No. 2, 1976. Printed in Great Britain. © 1976

Hamiltonian cosmology: a further investigation

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Received 30 June 1975, in final form 19 September 1975

Abstract. In a recent paper, Ryan claimed that Hamiltonian cosmology was dead. The present paper investigates the illness and makes some attempts at reviving the subject. It is found that in certain cases, the addition of another constraint to the variational principle will enable the correct field equations to be obtained. These results are then compared with some results for asymptotically flat spaces.

1. Introduction

As is well known the Einstein vacuum field equations can be derived from the variational principle

$$\delta \int R(\surd -g)\, d^4x = 0 \qquad (1.1)$$

where R is the scalar curvature of a four-dimensional Riemann space of signature +2. Arnowitt *et al* (1962, to be referred to as ADM) were able to cast this variational principle into Hamiltonian form. In this formalism the generalized coordinates, g_{ij}†, and momenta, π^{ij}, are functions of three space-like variables. In order to 'test' the usefulness of this procedure, several simple cases were investigated. These included the Friedmann universe (DeWitt 1967), homogeneous cosmologies (Misner 1969, Ryan 1972, involving the study of Hamiltonian cosmology) and cylindrical gravitational waves (Kuchar 1971).

The methods used by Misner (1969) and Ryan (1972) involved assuming a metric of the form

$$g_{ij} = g_{ab}(t)\sigma^a{}_i\sigma^b{}_j \qquad (1.2)$$

where the $\boldsymbol{\sigma}^a$ are three time-independent one-forms that satisfy

$$d\boldsymbol{\sigma}^a = C^a{}_{bc}\boldsymbol{\sigma}^b \wedge \boldsymbol{\sigma}^c \qquad (1.3)$$

and where the $C^a{}_{bc}$ are the structure constants. Also, π^{ij} is assumed to have a similar form. These assumptions can be used in the variational principle (1.1) (or its Hamiltonian equivalent) and the new generalized coordinates and momenta (g_{ab} and π^{ab}) are now discrete variables. However, as was first noted by Hawking (1969), the resultant variational principle does not always give the correct field equations.

MacCallum and Taub (1972) investigated this difficulty and found that the presence of spatial divergence terms (whose variation did not vanish) meant that in some cases

† Greek indices $\mu, \nu \ldots$ run from 0–3; Latin indices $i, j \ldots$ run from 1–3 and refer to coordinate frames; Latin indices $a, b \ldots$ run from 1–3 and refer to non-coordinate frames.

two of the equations differed from the usual field equations. They also attempted to derive these equations by other means. Ryan (1974) also discussed this problem and concluded that the trouble arose because the variation was performed in a non-coordinate frame. In the present paper an error, first noted by Misner (see reference 11 of Ryan 1974), in Ryan's work is corrected and it is shown that the variational principle is valid in non-coordinate frames. It is also shown that only one of the equations obtained is incorrect, and that this can be derived separately from the constraints of the system.

In § 2, the ADM formalism is briefly described and the reduction to homogeneous form is carried out. It is shown that whereas the variational principle works for non-coordinate frames, the requirement of spatial homogeneity prevents a boundary term being set equal to zero (as Hawking 1969 and MacCallum and Taub 1972 found). Some notation is introduced in § 3, and changes in the variational principle are discussed. Section 4 looks at the Poisson brackets of the constraints and suggests the introduction of a coordinate constraint, which will in general be non-holonomic, in order to derive the correct field equations. A preliminary investigation is made to determine the circumstances under which this constraint is holonomic. In § 5 this work is compared with work of Regge and Teitelboim (1974) on asymptotically flat spaces.

2. The variational principle in homogeneous spaces

In the ADM formalism the variational principle (1.1) is written as

$$\delta\int[-g_{ij}\pi^{ij}{}_{,0}-N\mathscr{H}-N_i\mathscr{H}^i-2(\pi^{ij}N_j-\tfrac{1}{2}\pi N^i+N^{|i}\sqrt{g})_{|i}]\,\mathrm{d}^3x\,\mathrm{d}t=0 \qquad (2.1)$$

where

$$\begin{aligned} &g_{ij}={}^4g_{ij}, && N=(-{}^4g^{00})^{-1/2},\\ &N_i={}^4g_{0i}, && \pi^{ij}=(-{}^4g)^{1/2}({}^4\Gamma^0_{pq}-g_{pq}{}^4\Gamma^0_{rs}g^{rs})g^{ip}g^{jq},\\ &\pi=\pi^{ij}g_{ij} \end{aligned} \qquad (2.2)$$

and g^{ij} is the inverse of g_{ij}. In this formalism N and N_i are the Lagrange multipliers corresponding to the constraints

$$\mathscr{H}=g^{-1/2}(\pi^{ij}\pi_{ij}-\tfrac{1}{2}\pi^2)-g^{1/2}({}^3R) \qquad (2.3)$$

and

$$\mathscr{H}^i=-2\pi^{ij}{}_{|j}. \qquad (2.4)$$

${}^3R$ is the scalar curvature derived from g_{ij} and the bar denotes covariant differentiation with respect to the g_{ij}. All indices are raised and lowered with g^{ij} and g_{ij}. Variation with respect to π^{ij}, g_{ij}, N and N_i gives the field equations as

$$g_{ij,0}=2Ng^{-1/2}(\pi_{ij}-\tfrac{1}{2}g_{ij}\pi)+2N_{(i|j)} \qquad (2.5a)$$

$$\begin{aligned}\pi^{ij}{}_{,0}=&-Ng^{1/2}({}^3R^{ij}-\tfrac{1}{2}g^{ij}\,{}^3R)+\tfrac{1}{2}Ng^{-1/2}g^{ij}(\pi^{mn}\pi_{mn}-\tfrac{1}{2}\pi^2)-2Ng^{-1/2}(\pi^{im}\pi_m{}^j-\tfrac{1}{2}\pi\pi^{ij})\\ &+g^{1/2}(N^{|ij}-g^{ij}N_{|m}{}^{|m})+(\pi^{ij}N^m)_{|m}-N^i{}_{|m}\pi^{mj}-N^j{}_{|m}\pi^{mi}\end{aligned} \qquad (2.5b)$$

$$\mathscr{H}=0. \qquad (2.5c)$$

$$\mathscr{H}^i=0. \qquad (2.5d)$$

spatial divergence in the integrand plays no part in the variational principle and can be neglected.

The metric is assumed to be spatially homogeneous (i.e. there exists a three-parameter isometry group which is transitive on a family of space-like hypersurfaces†) is

$$g_{ij} = g_{ab}(t)\sigma^a{}_i\sigma^b{}_j \tag{2.6}$$

where σ^a are independent of t and satisfy

$$\mathrm{d}\boldsymbol{\sigma}^a = C^a{}_{bc}\boldsymbol{\sigma}^b \wedge \boldsymbol{\sigma}^c. \tag{2.7}$$

$C^a{}_{bc}$ are the structure constants, and can be assumed to have one of the nine canonical forms found by Bianchi (1897). Also, $\pi^{ij} = (\det \sigma^a{}_i)\pi^{ab}(t)\sigma_a{}^i\sigma_b{}^j$ where $\sigma^a{}_i\sigma_b{}^i = \delta^a{}_b$. The first factor on the right-hand side appears because π^{ij} is a tensor density. The spatial integration in equation (2.1) can now be performed and the variational principle becomes

$$\delta \int [-g_{ab}\pi^{ab}{}_{,0} - N\mathcal{H} - N_a\mathcal{H}^a - 2(\pi^{ab}N_b - \tfrac{1}{2}\pi N^a)_{|a}]\,\mathrm{d}t = 0 \tag{2.8}$$

where $N_a(t) = N_i\sigma_a{}^i$ and $N = N(t)$. One point that is immediately apparent is that the variation of the spatial divergence term does not always vanish. In fact

$$(\pi^{ab}N_b - \tfrac{1}{2}\pi N^a)_{|a} = (\pi^{ab}N_b - \tfrac{1}{2}\pi N^a)C^c{}_{ca}. \tag{2.9}$$

Unless $C^c{}_{ca} = 0$, variation with respect to N_a will result in incorrect constraints, and so it seems that in Hamiltonian cosmology, this term should be removed from the variational principle, which now becomes

$$\delta \int (-g_{ab}\pi^{ab}{}_{,0} - N\mathcal{H} - N_a\mathcal{H}^a)\,\mathrm{d}t = 0. \tag{2.10}$$

Equation (2.10) still leads to incorrect field equations. As several authors (Hawking 1969, MacCallum and Taub 1972, Ryan 1974) have noted, the trouble arises when the variation of ${}^3R\sqrt{g}$ is taken with respect to g_{ab}. For an arbitrary vector basis

$$R_{bd} = \Gamma^c_{bd,c} - \Gamma^c_{bc,d} + \Gamma^c_{gc}\Gamma^g_{bd} - \Gamma^g_{bc}\Gamma^c_{dg}, \tag{2.11}$$

where

$$\Gamma^d{}_{ba} = \tfrac{1}{2}g^{dc}(C_{cab} + C_{bca} - C_{abc} + g_{ca,b} + g_{bc,a} - g_{ab,c}) \tag{2.12}$$

is the connection, and one finds

$$\delta\int R_{ab}g^{ab}\sqrt{g}\,\mathrm{d}V$$

$$= -\int R^{ab}\,\delta g_{ab}\sqrt{g}\,\mathrm{d}V + \tfrac{1}{2}\int g^{ab}\,{}^3R\sqrt{g}\,\delta g_{ab}\,\mathrm{d}V + \int g^{ab}\,\delta R_{ab}\sqrt{g}\,\mathrm{d}V \tag{2.13}$$

and

$$g^{ab}\,\delta R_{ab} = \omega^a{}_{|a} \tag{2.14}$$

$$\omega^a = g^{db}\,\delta\Gamma^a_{bd} - g^{ab}\,\delta\Gamma^c_{bc} = (g^{ad}g^{cb} - g^{ab}g^{dc})(\delta g_{cd})_{|b}. \tag{2.15}$$

† This definition does not include the models of Kantowski and Sachs (1966). These models have the four-parameter isometry group $G_4 = G_1 \otimes SO_3$ acting on a family of space-like hypersurfaces. However, this G_4 admits no transitive G_3.

Equation (2.14) is the corrected version of equation (2.9) of Ryan (1974) which has an extra term, $-C^a{}_{ab}\,\delta\Gamma^b{}_{cd}g^{cd}$, on the right-hand side. The possibility of an error in this equation was first noted in reference 11 of Ryan (1974). Now, provided δg_{ab} and $(\delta g_{ab})_{,c}$ can be made to vanish on the boundary, the last term in equation (2.13) will vanish and the variation will give the usual result. However, if g_{ab} and δg_{ab} are to be constants (or rather functions of time only) these conditions cannot be satisfied without δg_{ab} vanishing everywhere. This is the situation in Hamiltonian cosmology. In this case

$$\int \omega^a{}_{|a}\sqrt{g}\,\mathrm{d}V=\int \omega^a C^b{}_{ba}\sqrt{g}\,\mathrm{d}V \tag{2.16}$$

and will not vanish unless

$$C^b{}_{bd}(C^e{}_{ca}g^{dc}g^{af}-C^c{}_{ac}g^{ae}g^{df})\,\delta g_{ef}=\omega^a C^b{}_{ba}=0. \tag{2.17}$$

MacCallum and Taub (1972) showed that if equation (2.17) is to hold for arbitrary δg_{ab} then

$$C^b{}_{ba}=0. \tag{2.18}$$

Ellis and MacCallum (1969) called such cosmologies class A models. They include types I, II, VII_0, VI_0, IX and VIII in the Bianchi–Behr classification (Estabrook *et al* 1968) whereas models whose isometry group has type V, IV, VIII_h or VI_h are called class B models. Equation (2.10) will thus provide the correct field equations for class A models but not for class B models.

The equations obtained from equation (2.10) are

$$g_{ab,0}=K_{ab}-4N_{(a}a_{b)} \tag{2.19a}$$

$$\pi^{ab}{}_{,0}=L^{ab}-N\sqrt{g}(4a^a a^b+2a^c C^{(ab)}{}_c)-2N^{(a}C_c{}^{b)}{}_d\pi^{cd}+2\pi^{ab}N^c a_c \tag{2.19b}$$

$$\mathcal{H}=0 \tag{2.19c}$$

$$\mathcal{H}^a=0 \tag{2.19d}$$

where K_{ab} and L^{ab} are the frame components of the right-hand sides of equations (2.5*a*) and (2.5*b*) and

$$a_a=\tfrac{1}{2}C^c{}_{ac}. \tag{2.20}$$

The additional term $-N\sqrt{g}(4a^a a^b+2a^c C^{(ab)}{}_c)$ in equation (2.19*b*) results from the non-vanishing divergence in equation (2.13) and the term $2\pi^{ab}N^c a_c$ is simply (4.5) of MacCallum and Taub (1972). The term $-2N^{(a}C_c{}^{b)}{}_d\pi^{cd}$ does not appear explicitly in their work as they effectively considered N^a (rather than N_a) to be the independent variables, but appears implicitly in (4.4) of that paper. The extra term in equation (2.19*a*) does not seem to have been commented upon before. It can be eliminated by a different choice of Lagrangian, but if this were done, the constraints derived from the variational principle would be incorrect.

The constraints imply that each of these terms will be zero for class A models. For class B models, each term (apart from $-N\sqrt{g}(4a^a a^b+2a^c C^{(ab)}{}_c)$) will be zero only if $N_a=0$. Thus, for class B models, the available coordinate freedom has to be used to put $N_a=0$, which is the usual assumption in Hamiltonian cosmology.

3. Attempts to change the variational principle

It may be possible to somehow modify the variational principle in order to obtain the correct field equations, for which purpose it is convenient to use the following notation:

$$m^{ab}=\tfrac{1}{2}C^{(a}{}_{cd}\epsilon^{b)cd} \tag{3.1}$$

$$A^2=a^a a_a. \tag{3.2}$$

Then

$$C^a{}_{bc}=m^{ad}\epsilon_{dbc}+2a_{[b}\delta^a{}_{c]}. \tag{3.3}$$

Jacobi's identity, $C^c{}_{[ab}C^e{}_{d]c}=0$, implies

$$m^{ab}a_b=0. \tag{3.4}$$

For class B models, the metric can be written

$$g_{ab}=X_aX_b+Y_aY_b+A^{-2}a_aa_b \tag{3.5}$$

where X_a, Y_a and $A^{-1}a_a$ form an orthonormal basis. The seven parameters (X_a, Y_a, A) may be reduced to six by the inclusion of an additional constraint. In particular, the constraint

$$(X_bX_c-Y_bY_c)m^{bc}=0 \tag{3.6}$$

will ensure that the vectors $X_a\pm Y_a$ are eigenvectors of m^{ab}.

It is clear from equation (2.17) that, provided only certain variations δg_{ab} are allowed in the variational principle, the correct equations will be obtained. All that is needed is that

$$(2a^aa^b+a^cC^{(ab)}{}_c)\,\delta g_{ab}=0. \tag{3.7}$$

Equation (3.7) will be satisfied by choosing δg_{ab} to be of the form

$$\begin{aligned}\delta g_{ab}=\alpha(t)g_{ab}+\beta(t)a_{(a}X_{b)}+\gamma(t)a_{(a}Y_{b)}+\delta(t)X_{(a}Y_{b)}\\ +\sigma(t)[A^3X_aX_b-A^3Y_aY_b-\tfrac{1}{2}g^{-1/2}(\lambda_+-\lambda_-)a_aa_b]\end{aligned} \tag{3.8}$$

where $\lambda_\pm=\tfrac{1}{2}(X_a\pm Y_a)(X_b\pm Y_b)m^{ab}$. Thus it seems that for $N_a=0$, five components of the equations for $\pi^{ab}{}_{,0}$ are given correctly by the variational principle (2.10), which is one more than MacCallum and Taub claim. Furthermore, the remaining component can be deduced from the constraint $a_a\mathscr{H}^a=0$, using equation (2.19a) with $N_a=0$.

At this stage, it is important to note that when $a_aN^a=0$, equation (2.19a) together with the constraints implies that $a_a(g^{ab}{}_{,0})_{|b}=0$, so that

$$(2a^aa^b+a^cC^{(ab)}{}_c)g_{ab,0}=0. \tag{3.9}$$

Thus, if $a_aN^a=0$, the change δg_{ab} in the metric between one space-like hypersurface and a neighbouring space-like hypersurface will always satisfy equation (3.7). In the variational principle, it should not be necessary to consider variations in the metric other than those in equation (3.8). If this could be arranged, the incorrect equation for $\pi^{ab}{}_{,0}$ would not appear, but the corrected equation could be derived from the constraint $a_a\mathscr{H}^a=0$. It turns out that such a procedure will, in most cases, involve introducing a non-holonomic constraint. Before discussing this point it is convenient to look at the Poisson brackets of the constraints.

4. Addition of a constraint

For the variational principle $\delta \int (-g_{ab}\pi^{ab}{}_{,0} - N\mathscr{H} - N_a\mathscr{H}^a)\,\mathrm{d}t = 0$ with the total Hamiltonian $H = N\mathscr{H} + N_a\mathscr{H}^a$, the evolution equations of any function (f) of g_{ab} and π^{ab} can be written as

$$f_{,0} = [f, H] \tag{4.1}$$

where the Poisson bracket (Pb) is defined by

$$[f, g] = \frac{\partial f}{\partial g_{ab}} \frac{\partial g}{\partial \pi^{ab}} - \frac{\partial f}{\partial \pi^{ab}} \frac{\partial g}{\partial g_{ab}}. \tag{4.2}$$

In order that the constraints be preserved in time it is necessary that

$$[\mathscr{H}, H] \approx [\mathscr{H}^a, H] \approx 0. \tag{4.3}$$†

In the full ADM formalism (and in class A models) all of the constraints have weakly vanishing Pb's with each other and so equations (4.3) do not place any restriction on N and N_a. In class B models however, the Pb's do not all vanish, since

$$[\mathscr{H}, \mathscr{H}^c] \approx -2(\sqrt{g})a^c[3A^2 + g^{-1}(\lambda_+ - \lambda_-)^2] \tag{4.4a}$$

$$[\mathscr{H}^a, \mathscr{H}^b] \approx 0. \tag{4.4b}$$

Therefore, since N must be non-zero, equation (4.3) cannot be satisfied. Whereas $X_a\mathscr{H}^a$ and $Y_a\mathscr{H}^a$ both have weakly vanishing Pb's with the other constraints, $a_a\mathscr{H}^a$ has a non-vanishing Pb with $\mathscr{H}$.

Since there is some freedom in choosing the generalized coordinates and momenta (i.e. the usual coordinate freedom), it may be possible to add a new constraint ($\mathscr{C} = 0$) with its Lagrange multiplier, λ, so that the new Hamiltonian

$$H' = H + \lambda\mathscr{C} \tag{4.5}$$

would satisfy

$$[\mathscr{H}, H'] \approx [\mathscr{H}^a, H'] \approx [\mathscr{C}, H'] \approx 0 \tag{4.6}$$

for a certain choice of $N \neq 0$, N^a and λ. Such a constraint would be a coordinate condition. Provided $[a_a\mathscr{H}^a, \mathscr{C}]$ does not vanish weakly, equation (4.6) can then be satisfied. It would of course be desirable to choose a condition which would imply

$$a^aN_a = 0 \tag{4.7}$$

and it would be best if this constraint had the form $f(g_{ab}) = 0$. Now equation (4.7) is equivalent to (3.9), i.e.

$$(2a^aa^b + a^cC^{(ab)}{}_c)g_{ab,0} = 0. \tag{3.9}$$

If this expression can be integrated, the result would be a suitable coordinate condition to use, and would effectively ensure that any variation δg_{ab} in the metric would satisfy equation (3.7).

In type V spaces, where $m^{ab} = 0$, equation (3.9) can be written

$$(gA^6)_{,0} = 0. \tag{4.8}$$

† Following Dirac (1964), two quantities A, B are said to be weakly equal ($A \approx B$) if their difference vanishes when the field equations are satisfied.

Thus the constraint is holonomic. The constant of integration can be taken as unity and the constraint becomes

$$\mathscr{C} = gA^6 - 1 = 0. \tag{4.9}$$

If $m^a{}_a \neq 0$, equation (3.9) is not integrable for arbitrary $g_{ab,0}$, but for these types, it may be possible to place some additional restriction on the g_{ab} so that equation (3.9) will be integrable for the restricted metrics. Such a result is to be expected in view of the fact that MacCallum (1971) has found a variational principle for class B models in which $m^a{}_a = 0$.

One method for investigating this possibility is to proceed as follows: m^{ab} can be written as

$$m^{ab} = Xb^a b^c + Yc^a c^b \tag{4.10}$$

where X, Y, b^a and c^a are constants, with $b^a a_a = c^a a_a = 0$. Note that equation (4.10) does not determine X, Y, b^a or c^a uniquely. Having made a choice of b^a and c^a, the metric g_{ab} can be parametrized by the six numbers $\underline{b}\,.\,\underline{b}$, $\underline{b}\,.\,\underline{c}$, $\underline{c}\,.\,\underline{c}$ (i.e. $b^a b_a$ etc), and a^c, the contravariant components of a_c. Then $A^2 = a^c a_c$ and

$$g^{ab} = h^{ab} + A^{-2} a^a a^b \tag{4.11}$$

where

$$h^{ab} = [\underline{b}\,.\,\underline{b}\,\underline{c}\,.\,\underline{c} - (\underline{b}\,.\,\underline{c})^2]^{-1}(\underline{c}\,.\,\underline{c}\, b^a b^b - 2\underline{c}\,.\,\underline{b}\, b^{(a} c^{b)} + \underline{b}\,.\,\underline{b}\, c^a c^b). \tag{4.12}$$

Using

$$a^c \epsilon_{dcb} = S(\sqrt{g})^{-1} b_{[d} c_{b]}, \tag{4.13}$$

where $S = 2A[\underline{b}\,.\,\underline{b}\,\underline{c}\,.\,\underline{c} - (\underline{b}\,.\,\underline{c})^2]^{-1}$, it can be shown that

$$\tfrac{1}{2} m_e{}^d \epsilon_{dcb} g^{ab}{}_{,0}$$

$$= a^c h_{ae} m^{ed} \epsilon_{dcb} h^{ab}{}_{,0} = (2\sqrt{g})^{-1} S(\underline{b}\,.\,\underline{c})^2 [X\underline{b}\,.\,\underline{b}/\underline{b}\,.\,\underline{c} - Y\underline{c}\,.\,\underline{c}/\underline{b}\,.\,\underline{c}]_{,0} \tag{4.14}$$

if $\underline{b}\,.\,\underline{c} \neq 0$. (The case where $\underline{b}\,.\,\underline{c} = 0$ can be treated separately.) It follows that

$$(a_a a_b + a_c C_{(ab)}{}^c) g^{ab}{}_{,0} = (2\sqrt{g}A)^{-1}[(\sqrt{g}A^3)_{,0} - S(\underline{b}\,.\,\underline{c})^2 A (Xx - Yy)_{,0}] \tag{4.15}$$

where $x = \underline{b}\,.\,\underline{b}/\underline{b}\,.\,\underline{c}$ and $y = \underline{c}\,.\,\underline{c}/\underline{b}\,.\,\underline{c}$. Equation (4.13) can be used to show that $(\sqrt{g})^{-1}(\sqrt{g})_{,0} = -S^{-1} S_{,0}$, so

$$\sqrt{g} = kS^{-1} \tag{4.16}$$

where k is a constant that transforms as a scalar density under a transformation of base. Therefore, equation (3.9) can be written

$$k(\ln K)_{,0} - 4(1 - xy)^{-1}(Xx - Yy)_{,0} = \theta, \tag{4.17}$$

where $K = S^{-1} A^3$.

Now a one-form $\mathrm{d}f + g\,\mathrm{d}h$ can be written as $p\,\mathrm{d}q$ (i.e. is proportional to a differential) if and only if $\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h = 0$. Therefore equation (4.17) is integrable if and only if

$$(Xx + Yy)\, \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y = 0. \tag{4.18}$$

The case $Xx + Yy = 0$ corresponds to $m^a{}_a = 0$, which is the case found by MacCallum (1971). If $m^a{}_a \neq 0$, equations (4.17) and (4.18) show that $y = y(x)$ and $K = K(x)$, i.e.

there are two functional relations connecting x, y and K. If $y(x)$ is given, equations (4.17) will specify K to within a constant. The Hamiltonian for these cases will be

$$H' = H + \lambda \mathscr{C} + \sigma \mathscr{D} \tag{4.19}$$

with

$$\mathscr{C} = K - K(x) \tag{4.20a}$$

$$\mathscr{D} = y - y(x). \tag{4.20b}$$

If the metric is such that b^a and c^a can be chosen to satisfy $\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$ consistently, equation (3.9) will once again be integrable. However, $\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$ at all times will again impose further conditions on the metric. A number of possibilities arise, one of which is the case $m^a{}_a = 0$. The Hamiltonian will again be given by equation (4.19) with

$$\mathscr{C} = gA^3 - 1 \tag{4.21a}$$

$$\mathscr{D} = \underset{\sim}{b} \cdot \underset{\sim}{c}. \tag{4.21b}$$

For those cases in which equation (3.9) is not integrable a non-holonomic constraint has to be used. It is not certain how to incorporate such a constraint into a Hamiltonian formalism. One possibility is to rewrite Hamilton's equations as

$$g_{ab,0} = \partial H / \partial \pi^{ab} \tag{4.22a}$$

$$\pi^{ab}{}_{,0} = -(\partial H / \partial g_{ab}) - \lambda C^{ab} \tag{4.22b}$$

where C^{ab} is the coefficient of $g_{ab,0}$ in equation (3.9) and the value of λ is fixed by $a_a \mathscr{H}^a = 0$. This is essentially the procedure suggested by Ryan (1974). It is unclear how equations (4.22) can be derived from a variational principle, let alone how to proceed with the canonical quantization of these models.

5. The variational principle in asymptotically flat spaces

In a recent paper, Regge and Teitelboim (1974) discussed the variational principle (1.1) in the case of asymptotically flat metrics, where the spatial boundary of the region of integration is at spatial infinity. They found initially that in order to obtain the correct field equations (or indeed any meaningful field equations at all) from

$$g_{ij,0} = \delta H / \delta \pi^{ij} \tag{5.1a}$$

$$\pi_{ij,0} = -\delta H / \delta g_{ij} \tag{5.1b}$$

it is necessary to write

$$H = \int N\mathscr{H} + N_i \mathscr{H}^i \, \mathrm{d}^3x + E \tag{5.2}$$

which differs from the usual choice by the surface integral

$$E = \oint (g_{ik,i} - g_{ii,k}) \, \mathrm{d}^2 S_k. \tag{5.3}$$

Briefly their reasoning is as follows. Near spatial infinity the metric can be written in the form

$$ds^2 \underset{r\to\infty}{\sim} -\left(1-\frac{M}{8\pi r}\right)dt^2+\left(\delta_{ij}+\frac{M}{8\pi}\frac{x^i x^j}{r^3}\right)dx^i\,dx^j. \tag{5.4}$$

It turns out that for any change δg_{ij} in the metric that preserves this form

$$\delta\int N\mathcal{H}+N_i\mathcal{H}^i\,d^3x=\int A^{ij}\,\delta g_{ij}-\delta\{E[g_{ij}]\} \tag{5.5}$$

where A^{ij} is the right-hand side of equation (2.5*b*). Since for the metric (5.4), $E[g_{ij}]=M$, its variation will not in general vanish. Thus if equations (5.1) are to give the correct field equations, H must be chosen as in equation (5.2).

It is interesting to note the similarities between this result and the work on Hamiltonian cosmology. In both cases the boundary of the region of integration is in a region where the space has some symmetry. (In the case of asymptotically flat spaces, the group is $C_1\otimes SO_3$ where C_1 is a one-parameter conformal motion.) In this region the metric must be of a certain type, and any variation in the metric must preserve this type. Whereas it is true that in asymptotically flat spaces, the most general variation δg_{ij} that preserves the form of the metric will have $\delta M\neq 0$, the most general variation is not essential for a variational principle. In fact, it is usual to suitably restrict δg_{ij} and $\delta g_{ij,k}$ on the boundary of the region of integration, and in the present example it does not seem unreasonable to restrict δg_{ij} and $\delta g_{ij,k}$ on the boundary to satisfy $\delta E=0$.

In Hamiltonian cosmology, δg_{ab} can be restricted on the boundary (and hence everywhere) to satisfy $(2a^a a^b+C^{(ab)}{}_c a^c)\,\delta g_{ab}=0$. In the same way that $a_a N^a=0$ guarantees $(2a^a a^b+C^{(ab)}{}_c a^c)g_{ab,0}=0$ for any solution of the field equations, $E_{,0}=0$ can be deduced from the equations for asymptotically flat spaces; this means that any solution of these equations will satisfy the constraint

$$E=\text{constant}. \tag{5.6}$$

If this constraint is included in the variational principle in the usual way, the difficulties encountered by Regge and Teitelboim will be avoided. This is similar to the approach adopted by Regge and Teitelboim who add to the Hamiltonian the term $\alpha^{\perp}(p_{\perp}-P_{\perp})$ where $\alpha^{\perp}$ is a Lagrange multiplier describing time-like translations at infinity and $P_{\perp}=E$.

6. Conclusions

It appears that whenever the Einstein variational principle is used, care should be taken to ensure that the correct field equations are obtained. Certainly in the case of homogeneous spaces of class B the usual variational principle breaks down in a number of places. By neglecting certain terms in the Lagrangian, some of these difficulties can be overcome, but there still remains the problem that the field equations for $\pi^{ab}{}_{,0}$ are incorrect. The incorrect terms arise from a surface integral that does not vanish for certain variations δg_{ab}, though it is possible to include a constraint that will prohibit these variations. Unfortunately, for most class B models, the necessary constraint is non-holonomic. Only for type V models is it holonomic for arbitrary g_{ab}, although for other class B models it may be so if g_{ab} satisfies some additional constraint (e.g.

$m^a{}_a = 0$). The Einstein field equations for these models can therefore be derived from a variational principle. The Hamiltonian for type V models is given by equation (4.3) with $\mathscr{C}$ given by equation (4.9). The Hamiltonian for other class B metrics for which equation (3.9) is integrable will be given by equation (4.19) with $\mathscr{C}$ and $\mathscr{D}$ given by equations (4.20) or (4.21). A similar problem arises for asymptotically flat spaces, but in this case also it is possible to introduce a (holonomic) constraint that will suitably restrict the variations δg_{ij}.

Acknowledgments

The author thanks Dr C B G McIntosh for his help and encouragement during the preparation of this work and Dr M A H MacCallum for his comments on an earlier draft of this paper.

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